Quasi-categories — or how I learned to stop worrying ~~abo~~ and love simplicial sets.

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§ (weak) Kan complexes – for whom the horn lifts:

<u>Def'n</u>: k^{th} horn $\Lambda^n_k \subset \Delta^n$ sub-simpl. set of all proper faces of Δ^n but the k^{th}

<u>Ex</u>: $|\Delta^2| =$ 

$|\Lambda^2_0| =$ 

$|\Lambda^2_1| =$ 

$|\Lambda^2_2| =$ 

<u>Def'n</u> A <u>Kan complex</u> $S_\cdot$ is a simplicial set with the following horn-lifting property:

$\forall\ 0 \le i \le n$:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & S \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

<u>Ex</u>: $X \in$ (cpctly gen. Hausdorff) space

$\Rightarrow$ Sing X is a Kan complex:

there is a deformation retract, compose with this

$$\begin{tikzcd} \Lambda \arrow[r] & \mathrm{Sing}\, X \\ \Delta \arrow[u] \arrow[ur, dashed] & \end{tikzcd}$$

<u>Remark</u>: S is fibrant in $sSet_{Quillen}$ $\Leftrightarrow$ S is Kan complex

§2. quasi-category theory from the quasi-mathematician

Def'n: $S \in sSet$. $V_0 \subset S_0$

$S_{V_0} \subset S$ is the sub-simplicial set with n-simplices

st $\forall\, \Delta^0 \to \Delta^n$,
$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[rr, bend right] & \Delta^n \arrow[r] & S \end{tikzcd}$$
$\in V_0$

$(S_{V_0})_n = \{\sigma \in S_n \text{ st. } \uparrow \}$

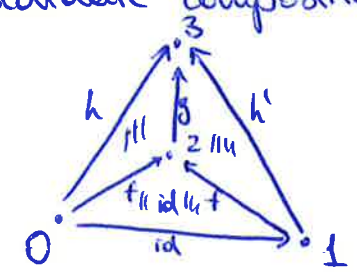
If $\mathcal{C} \in q\text{-Cat}$, then $\mathcal{C}_{V_0}$ is the full sub-quasi-category spanned by V_0.

Def'n: $\mathcal{C} \in q\text{Cat}$, f, g morphisms $f, g : X \to Y$
(i.e. $f, g \in \mathcal{C}_1$ st $d_0 f = d_0 g = \cancel{X}$, $d_1 f = d_1 g = X$)

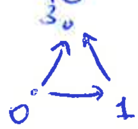
A homotopy from f to g (between f and g) is a 2-simplex $\cancel{\eta}$ $\eta \in \mathcal{C}_2$

$$\begin{tikzcd} & X^{1} \arrow[dr, "g"] & \\ {}_{0}X \arrow[ur, "id_X"] \arrow[rr, "f"'] \arrow[rr, phantom, shift left=3, "\Uparrow\eta"] & & Y_{2} \end{tikzcd}$$

Lemma All candidate compositions are homotopic:



now use the horn lifting condition to fill in the side



Def'n A morphism in a quasi-category $\mathcal{C}$ $f:\Delta^1 \to \mathcal{C}$ is an <u>equivalence</u> if $[f]$ is an isomorphism in $\mathrm{Ho}\,\mathcal{C}$.

Prop: A quasi-category $\mathcal{C}$ is a Kan complex iff $\mathrm{Ho}\,\mathcal{C}$ is a groupoid.

Def'n $\mathcal{C}, \mathcal{D}$ ordinary categories. The <u>join</u> $\mathcal{C} * \mathcal{D}$ is the following category:

$$\mathrm{ob}(\mathcal{C} * \mathcal{D}) = \mathcal{C} \amalg \mathcal{D}$$

$$(\mathcal{C} * \mathcal{D})(X,Y) = \begin{cases} \mathcal{C}(X,Y) & \text{if } X,Y \in \mathrm{ob}\,\mathcal{C} \\ \mathcal{D}(X,Y) & \text{if } X,Y \in \mathrm{ob}\,\mathcal{D} \\ * & \text{if } X \in \mathrm{ob}\,\mathcal{C},\ Y \in \mathrm{ob}\,\mathcal{D} \\ \emptyset & \text{if } X \in \mathrm{ob}\,\mathcal{D},\ Y \in \mathrm{ob}\,\mathcal{C} \end{cases}$$

Example: $\mathcal{C} \in \mathrm{Cat}, \mathcal{D} = \mathbb{1}$ $\quad \mathcal{C} * \mathbb{1}$ is $\mathcal{C}$ with one new terminal object

Ex $\mathcal{C} = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$ $\quad \mathcal{C} * \mathbb{1} = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$ $\quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$

$\mathcal{C} = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$ $\quad \mathbb{1} * \mathcal{C}$ $/\!/$ $\quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$

Def'n $S, S' \in \mathrm{sSet}$ The <u>join</u> $S * S'$ is $\forall J \in \Delta$

$$(S * S')_J = \coprod_{\substack{J = I \cup I' \\ \text{st. } \forall i \in I \\ i' \in I', \\ i < i'}} S_I \times S_{I'}$$

$$(S * S')_n = S_n \cup \bigcup_{i+j=n-1} S_i \times S'_j \cup S'_n$$

Ex: $\Lambda^2_0 * \Delta^0 = \Delta^1 \times \Delta^1$

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] \arrow[dr] & (0,1) \arrow[d] \\ (1,0) \arrow[r] & (1,1) \end{tikzcd}$$

Def'n An object $x \in \mathcal{C}$ is a <u>terminal object</u> if

$$\mathcal{C}_{/x} \longrightarrow \mathcal{C} \quad \text{is an acyclic Kan fibration.}$$

Prop: $\mathcal{C} \in qCat$. $x \in \mathcal{C}$ terminal. Then $\forall y \in \mathcal{C}_0$ $\exists f : \Delta^1 \longrightarrow \mathcal{C}$ st $d_1(f) = x$, $d_0(f) = y$ unique up to homotopy.

(In other words, $\forall y \in \mathcal{C}_0$, $Map_{\mathcal{C}}(y, x)$ is contractible.)

Prop $\mathcal{C} \in Cat$, $c \in \mathcal{C}$.

Then c is terminal in $\mathcal{C}$ iff c is terminal in $N\mathcal{C}$.

Def'n $S \in sSet$, $\mathcal{C} \in qCat$, $p : S \to \mathcal{C}$.

The <u>colimit</u> of p is any initial object of $\mathcal{C}_{p/}$.

Def'n: $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{D} \longrightarrow \mathcal{C}$, $\mathcal{C}, \mathcal{D} \in qCat$

F is the <u>left adjoint</u> of G if

$$Map_{\mathcal{D}}(FC, D) \simeq Map_{\mathcal{C}}(C, GD)$$

$$u \in Fun(\mathcal{C}, \mathcal{C}) : \quad u : id_{\mathcal{C}} \longrightarrow GF$$